A short note on the curvature perturbation at second order

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Abstract. Working with perturbations about an FLRW spacetime, we compute the gauge-invariant curvature perturbation to second order solely in terms of scalar field fluctuations. Using the curvature perturbation on uniform density hypersurfaces as our starting point, we give our results in terms of field fluctuations in the flat gauge, incorporating both large and small scale behaviour. For ease of future numerical implementation we give our result in terms of the scalar field fluctuations and their time derivatives.

1. Introduction

The first detection of the anisotropies in the cosmic microwave background by the COBE satellite [1] paved the way for the predictions of the standard cosmological model to be tested. In the decades since, experiments have tremendously increased in their sophistication enabling us to put ever stronger constraints on our theory of the early universe [2, 3]. The most recent experimental success was the detection of B-mode polarization by the BICEP experiment [4]. Although a large fraction of this signal is now explained by galactic foregrounds [5], there is still room for a signal of primordial gravitational waves from inflation. This would be a huge success for inflationary cosmology.

The initial inhomogeneities in the density field are generated by quantum fluctuations during inflation. In the simplest models this is a single field slowly rolling down a potential, but there are a plethora of models with multiple fields, fields with complicated kinetic terms, non-minimally coupled fields, etc. (see e.g. Ref. [6]). In order to model the inhomogeneities in the universe we use cosmological perturbation theory. This involves taking the homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) solution as a background spacetime and adding small, inhomogeneous perturbations on top (e.g. [7, 8, 9]). The linear theory is a huge success, matching the CMB and large scale structure excellently, however given the quality of the data that we are now fortunate to obtain, we can use higher order theory to put more restrictive constraints on models, and aim to rule out large classes of models.

In this article we focus on one particular quantity: the curvature perturbation on uniform density hypersurfaces, ζ. This gauge invariant quantity is the variable of choice when computing, say, the bispectrum of inflationary fluctuations. We derive the expression giving the curvature perturbation at second order in perturbation theory, ζ_2, in terms of scalar field variables, allowing for multiple minimally coupled scalar fields. Previous studies have either focused on large scales or have included a limited number of fields, either using perturbation theory or the δN formalism (for a non-exhaustive list of references see Refs. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). We envisage our calculation to be of particular use for future numerical computations of bispectra or other inflationary parameters and, to that end, ensure that the resultant expression is in a closed form, containing only field fluctuations and their single time derivative.

The paper is structured as follows: in the next section we present the governing equations for perturbations up to second order. In Section 3 we define the curvature perturbation, and present our main result. Then, in Section 4, we specialise to the case of a single field in the slow-roll limit. We close with a discussion in Section 5. We assume an FLRW background spacetime with zero spatial curvature, and use conformal time, η, throughout. Greek indices, μ, ν, λ, cover the entire spacetime range, from $0, \dots 3$, while we use lower case Latin, i, j, k, to denote spatial indices running from $1, \dots 3$. Upper case Latin indices, I, J, K, denote different scalar fields.

2. Governing equations

In this section we review the governing equations for the cosmological perturbations up to second order that we will use when computing the curvature perturbation for a system containing multiple scalar fields. We follow the notation and definitions of, e.g., Refs. [22, 23] throughout. In this paper we consider on scalar perturbations to FLRW, for which the line element takes the form

$$ds^2 = a^2(\eta)\Big[-(1+2\phi)d\eta^2+2aB_{,i}dx^id\eta+\Big((1-2\psi)\delta_{ij}+2E_{,ij}\Big)dx^idx^i\Big]\,,\tag{2.1}$$

where perturbed quantities are then expanded, order by order, as, e.g., $\phi = \phi_1 + \frac{1}{2}\phi_2$. In the uniform curvature gauge, $E = \psi = 0$, and the two scalar degrees of freedom in the metric are ϕ and B.

2.1. The field equations

The governing field equations are obtained by perturbing the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, and truncating at the required order. In the background, we obtain the usual Friedmann and acceleration equations:

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left(\sum_K \frac{1}{2}{\varphi'_{0K}}^2 + a^2U_0\right)\,,\tag{2.2}$$

$$\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a} = 8\pi G\left(\sum_K \frac{1}{2}{\varphi'_{0K}}^2 - a^2U_0\right)\,.\tag{2.3}$$

In the following, we will now present the linear and second order field equations in the uniform curvature gauge.

2.1.1. First order

The $0-0$ component of the Einstein equations at first order is

$$2a^2U_0\phi_1 + \sum_K \varphi'_{0K}\delta{\varphi_{1K}}' + a^2\delta U_1 + \frac{\mathcal{H}}{4\pi G}\nabla^2 B_1 = 0\,,\tag{2.4}$$

where we have used the background equations from above to simply the expression. The $0-i$ part gives

$$\mathcal{H}\phi_1 - 4\pi G\sum_K \varphi'_{0K}\delta\varphi_{1K} = 0\,.\tag{2.5}$$

Although they are not needed in the following, for completeness we also give, from the spatial component of the Einstein equations, the trace free equation

$$B_1' + 2\mathcal{H}B_1 + \phi_1 = 0\,,\tag{2.6}$$

governing the evolution of the shear, and, using Eq. (2.6) and the background field equations, the first order trace in its simplest form is

$$\mathcal{H}\phi_1' + 4\pi G\left[a^2\delta U_1 + 2a^2U_0\phi_1 - \sum_K \varphi'_{0K}\delta{\varphi_{1K}}'\right] = 0\,.\tag{2.7}$$

Eq. (2.4) and Eq. (2.5) above can then be combined to give

$$\nabla^2 B_1 = -\frac{4\pi G}{\mathcal{H}} \sum_K \left(X_K \delta\varphi_{1K} + \varphi'_{0K} \delta\varphi_{1K}{}' \right) , \tag{2.8}$$

where, following Ref. [24], we have defined

$$X_K \equiv a^2 \left(\frac{8\pi G}{\mathcal{H}} U_0 \varphi'_{0K} + U_{,\varphi_K} \right) . \tag{2.9}$$

2.1.2. Second order Using the first order and background $0-0$ equations, Eqs. (2.2) and (2.4), the $0-0$ component of the second order Einstein equation is

$$\begin{aligned} 8\pi G a^2 U_0 \left(\phi_2 + B_{1,k} B_{1,}{}^k \right) + \mathcal{H} \nabla^2 B_2 + \frac{1}{2} \left[B_{1,kl} B_{1,}{}^{kl} - \left(\nabla^2 B_1 \right)^2 \right] - 2\mathcal{H} \phi_{1,k} B_{1,}{}^k \\ +4\pi G \sum_K \left[\varphi'_{0K} \delta\varphi_{2K}{}' + a^2 \delta U_2 + 4a^2 \delta U_1 \phi_1 + \delta\varphi_{1K}{}'^2 + \delta\varphi_{1K,k} \delta\varphi_{1K,}{}^k \right] = 0 , \end{aligned} \tag{2.10}$$

while the $0-i$ Einstein equation is given by

$$\begin{aligned} \mathcal{H}\phi_{2,i} - 4\mathcal{H}\phi_1\phi_{1,i} + 2\mathcal{H} B_{1,ki} B_{1,}{}^k + B_{1,ki}\phi_{1,}{}^k - \nabla^2 B_1 \phi_{1,i} \\ -4\pi G \sum_K \left[\varphi'_{0K} \delta\varphi_{2K,i} + 2\delta\varphi_{1K}{}' \delta\varphi_{1K,i} \right] = 0 . \end{aligned} \tag{2.11}$$

We then use Eq. (2.5), rewrite Eq. (2.11) and take the trace, to obtain

$$\begin{aligned} \mathcal{H} \left(\phi_2 - 2\phi_1^2 + B_{1,k} B_{1,}{}^k \right) - 4\pi G \sum_K \varphi'_{0K} \delta\varphi_{2K} + \nabla^{-2} \left(\phi_{1,kl} B_{1,}{}^{kl} - \nabla^2 B_1 \nabla^2 \phi_1 \right) \\ -8\pi G \sum_K \nabla^{-2} \left(\delta\varphi_{1K}{}' \nabla^2 \delta\varphi_{1K} + \delta\varphi'_{1K,l} \delta\varphi_{1K,}{}^l \right) = 0 , \end{aligned} \tag{2.12}$$

where we have introduced the inverse Laplacian, defined as $\nabla^{-2}(\nabla^2 X) = X$, and have utilised the following identities

$$\begin{aligned} \nabla^2 \left(\phi_1^2 \right) &= 2 \left(\phi_1 \nabla^2 \phi + \phi_{1,k} \phi_{1,}{}^k \right) , \\ \nabla^2 \left(B_{1,k} B_{1,}{}^k \right) &= 2 \left(B_{1,k} \nabla^2 B_{1,}{}^k + B_{1,kl} B_{1,}{}^{kl} \right) . \end{aligned} \tag{2.13}$$

2.2. The Klein-Gordon equation

The evolution of the scalar fields is given by the Klein-Gordon equation, obtained from energy-momentum conservation. In the background, this gives

$$\varphi''_{0I} + 2\mathcal{H}\varphi'_{0I} + a^2 U_{,\varphi_I} = 0 , \tag{2.14}$$

and for linear scalar field fluctuations is

$$\delta\varphi_{1I}{}'' + 2\mathcal{H}\delta\varphi_{1I}{}' + 2a^2 U_{,\varphi_I}\phi_1 - \nabla^2\delta\varphi_{1I} - \varphi'_{0I}\nabla^2 B_1 - \varphi'_{0I}\phi'_1 + a^2 \sum_K U_{,\varphi_I\varphi_K} \delta\varphi_{1K} = 0 . \tag{2.15}$$

The latter can then be written in a closed form, using the linear Einstein equations (see e.g. Ref. [24]), as

$$\begin{aligned} \delta\varphi_{1I}{}'' + 2\mathcal{H}\delta\varphi_{1I}{}' - \nabla^2\delta\varphi_{1I} \\ +a^2 \sum_K \left\{ U_{,\varphi_K\varphi_I} + \frac{8\pi G}{\mathcal{H}} \left(\varphi'_{0I} U_{,\varphi_K} + \varphi'_{0K} U_{,\varphi_I} + \varphi'_{0K}\varphi'_{0I} \frac{8\pi G}{\mathcal{H}} U_0 \right) \right\} \delta\varphi_{1K} = 0 . \end{aligned} \tag{2.16}$$

For completeness, we present the Klein-Gordon equation at second order in the Appendix.

3. The curvature perturbation

In order to arrive at meaningful calculations within cosmological perturbation theory, we have to consider gauge invariant quantities. There are several gauge invariant curvature perturbations, including the comoving curvature perturbation, $\mathcal{R}$ [25], the Newtonian curvature perturbation Ψ [7, 9] and the curvature perturbation on uniform density hypersurfaces, ζ, [26]. In this article we will focus on the latter, and in this section will present our major result of the paper and compute ζ in terms of scalar field fluctuations.

3.1. The curvature perturbation, ζ, in terms of the energy density

The curvature perturbation on uniform density hypersurfaces, ζ is given, at first order in terms of fluid variables as

$$-\zeta_1 \equiv \psi_1 + \mathcal{H}\frac{\delta\rho_1}{\rho_0'}\,, \tag{3.1}$$

which, when the RHS is evaluated in the uniform curvature gauge, simply gives

$$\zeta_1 = -\mathcal{H}\frac{\delta\rho_1}{\rho_0'}\,. \tag{3.2}$$

At second order the expression is more complicated and, expressed in the uniform curvature gauge, is

$$\begin{aligned} -\zeta_2 = {} & \frac{\mathcal{H}}{\rho_0'}\delta\rho_2 - 2\mathcal{H}\frac{\delta\rho_1\delta\rho_1'}{{\rho_0'}^2} - \mathcal{H}^2\left(5+3c_{\rm s}^2\right)\left(\frac{\delta\rho_1}{\rho_0'}\right)^2 + \frac{1}{2}\frac{\delta\rho_{1,k}\delta\rho_{1,}{}^{k}}{{\rho_0'}^2} + \frac{1}{\rho_0'}B_{1,k}\delta\rho_{1,}{}^{k} \\ & -\frac{1}{2}\nabla^{-2}\left\{\left[\frac{\delta\rho_{1,}{}^{i}\delta\rho_{1,}{}^{j}}{{\rho_0'}^2} + \frac{2}{\rho_0'}\delta\rho_{1,}{}^{(i}B_{1,}{}^{j)}\right]_{,ij}\right\}\,. \end{aligned} \tag{3.3}$$

Note that there are different definitions for ζ. In this work we follow the review by Malik & Wands [22], which can be related to the definition of ζ introduced by Salopek & Bond [27] through

$$\zeta_{\rm MW} = \zeta_{\rm SB} + \zeta_{\rm SB}^2\,. \tag{3.4}$$

3.2. Relating fluid to field variables

In this section we present the relationship between the fluid variables and scalar field variables. We simply quote the results here, which are obtained by comparing the energy momentum tensor components for a perfect fluid and a collection of minimally coupled scalar fields, see Refs. [24, 22].

In the background the relationships are

$$\rho_0 = \sum_K \frac{1}{2a^2}{\varphi_{0K}'}^2 + U_0\,, \tag{3.5}$$

$$P_0 = \sum_K \frac{1}{2a^2}{\varphi_{0K}'}^2 - U_0\,. \tag{3.6}$$

Considering now perturbations in the uniform curvature gauge, we find that the density perturbation at first order is in terms of the field fluctuations

$$\delta\rho_1 = \frac{1}{a^2}\sum_K \left(\varphi_{0K}'\delta\varphi_{1K}' - {\varphi_{0K}'}^2\phi_1\right) + \delta U_1\,, \tag{3.7}$$

while the equivalent expression at second order is

$$\begin{aligned}\delta\rho_2 = \frac{1}{a^2}\sum_K &\Big[\varphi_{0K}'\delta\varphi_{2K}' - 4\varphi_{0K}'\phi_1\delta\varphi_{1K}' - {\varphi_{0K}'}^2\phi_2 + 4{\varphi_{0K}'}^2\phi_1^2 + {\delta\varphi_{1K}'}^2 + a^2\delta U_2 \\ &+\delta\varphi_{1K,l}\delta\varphi_{1K,}{}^{l} - {\varphi_{0K}'}^2 B_{1,k}B_{1,}{}^{k}\Big] \\ -\frac{2}{a^2}\sum_K \varphi_{0K}'\delta\varphi_{1K,l}&\left[\frac{\sum_L \varphi_{0L}'\delta\varphi_{1L,}{}^{l}}{\sum_M {\varphi_{0M}'}^2} - \frac{4\pi G}{\mathcal{H}}\nabla^{-2}\sum_L\left(X_L\delta\varphi_{1L}+\varphi_{0L}'\delta\varphi_{1L}'\right)_{,}{}^{l}\right]\,,\end{aligned} \tag{3.8}$$

where we have defined

$$\delta U_1 = \sum_K U_{,\varphi_K}\delta\varphi_{1K}\,, \tag{3.9}$$

$$\delta U_2 = \sum_{K,L} U_{,\varphi_K\varphi_L}\delta\varphi_{1K}\delta\varphi_{1L} + \sum_K U_{,\varphi_K}\delta\varphi_{2K}\,. \tag{3.10}$$

The energy density perturbations can be written in terms of only scalar field quantities by using the Einstein field equations at first and second order. This gives

$$\delta\rho_1 = \frac{1}{a^2}\sum_K \varphi_{0K}'\Big(\delta\varphi_{1K}' - \varphi_{0K}'\frac{4\pi G}{\mathcal{H}}\sum_L \varphi_{0L}'\delta\varphi_{1L}\Big) + \delta U_1\,, \tag{3.11}$$

and

$$\begin{aligned}\delta\rho_2 = \frac{1}{a^2}\sum_{K,L,M}&\Bigg[\varphi_{0K}'\delta\varphi_{2K}' + a^2\delta U_2 + {\delta\varphi_{1K}'}^2 + \delta\varphi_{1K,l}\delta\varphi_{1K,}{}^{l} - \frac{16\pi G}{\mathcal{H}}\varphi_{0K}'\varphi_{0L}'\delta\varphi_{1K}'\delta\varphi_{1L} \\ -\frac{4\pi G}{\mathcal{H}}\varphi_{0K}'^2&\Bigg\{\varphi_{0L}'\delta\varphi_{2L} - \frac{8\pi G}{\mathcal{H}}\varphi_{0L}'\varphi_{0M}'\delta\varphi_{1L}\delta\varphi_{1M} + 2\nabla^{-2}\Big(\delta\varphi_{1L}'\nabla^2\delta\varphi_{1L} + \delta\varphi_{1L}'{}_{,l}\delta\varphi_{1L,}{}^{l}\Big) \\ -\frac{4\pi G}{\mathcal{H}^2}\nabla^{-2}&\Big(\varphi_{0M}'\nabla^2\delta\varphi_{1M}(X_L\delta\varphi_{1L}+\varphi_{0L}'\delta\varphi_{1L}') - \varphi_{0L}'\delta\varphi_{1L,kl}\nabla^{-2}(X_M\delta\varphi_{1M}+\varphi_{0M}'\delta\varphi_{1M}')_{,}{}^{kl}\Big)\Bigg\}\Bigg] \\ -\frac{2}{a^2}\sum_L \varphi_{0L}'\delta\varphi_{1L,l}&\left[\frac{\sum_K \varphi_{0K}'\delta\varphi_{1K,}{}^{l}}{\sum_M \varphi_{0M}'^2} - \frac{4\pi G}{\mathcal{H}}\nabla^{-2}\sum_K(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}')_{,}{}^{l}\right]\,.\end{aligned} \tag{3.12}$$

3.3. The curvature perturbation, ζ, in terms of the scalar fields

We now have the tools at hand to express the gauge-invariant curvature perturbation on uniform density hypersurfaces in terms of the scalar field fluctuations. Substituting Eq. (3.7) into Eq. (3.1), and using Eq. (2.5) to relate the metric perturbation ϕ_1 to the scalar fields as

$$\phi_1 = \frac{4\pi G}{\mathcal{H}}\sum_K \varphi_{0K}'\delta\varphi_{1K}\,, \tag{3.13}$$

we obtain

$$\zeta_1 = \frac{1}{a^2}\left[\frac{\sum_K\left(\varphi_{0K}'\delta\varphi_{1K}' + a^2U_{,\varphi_K}\delta\varphi_{1K}\right)}{3\sum_L {\varphi_{0L}'}^2} - \frac{\mathcal{H}\sum_K\varphi_{0K}'\delta\varphi_{1K}}{\sum_L {\varphi_{0L}'}^2 + 2a^2U_0}\right]\,. \tag{3.14}$$

At second order, the expression is again more complicated, but we follow the same procedure, by substituting for the metric perturbations from Sections 2.1.1 and 2.1.2 and the energy density perturbations in Eqs. (3.7) and (3.8) into Eq. (3.3). We also use the Klein Gordon equations in Section 2.2 to substitute for φ_{0K}'' and $\delta\varphi_{1K}''$. This results in the following expression for ζ_2 in terms of only scalar field perturbations

$$
\begin{aligned}
\zeta_2 = &\frac{1}{3\sum_N \varphi_{0N}'^2}\sum_K \Bigg[\varphi_{0K}'\delta\varphi_{2K}' - 4\delta\varphi_{1K}'Y_K + 2Y_K^2 - \frac{4\pi G}{\mathcal{H}}\varphi_{0K}'\sum_M \varphi_{0M}'\delta\varphi_{2M} \\
&-\frac{4\pi G}{\mathcal{H}^2}\varphi_{0K}'\nabla^{-2}\sum_M\Big(Y_{K,kl}\nabla^{-2}(X_M\delta\varphi_{1M}+\varphi_{0M}'\delta\varphi_{1M}')_{,}{}^{kl} - \nabla^2 Y_K(X_M\delta\varphi_{1M}+\varphi_{0M}'\delta\varphi_{1M}')\Big) \\
&-\frac{8\pi G}{\mathcal{H}}\varphi_{0K}'^2\sum_M\nabla^{-2}\Big(\delta\varphi_{1M}'\nabla^2\delta\varphi_{1M}+\delta\varphi_{1M,l}\delta\varphi_{1M}'{}_{,}{}^{l}\Big) + \delta\varphi_{1K}'^2 + a^2\delta U_2 + \delta\varphi_{1K,l}\delta\varphi_{1K,}{}^{l}\Bigg] \\
&-\frac{2}{3\sum_N\varphi_{0N}'^2}\sum_M\varphi_{0M}'\delta\varphi_{M,l}\left[\frac{\sum_K\varphi_{0K}'\delta\varphi_{1K,l}}{\sum_L\varphi_{0L}'^2} - \frac{4\pi G}{\mathcal{H}}\nabla^{-2}\sum_K\Big(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\Big)_{,}{}^{l}\right] \\
&+\frac{2}{9\mathcal{H}(\sum_P\varphi_{0P}'^2)^2}\sum_{K,M,N}\Bigg[\nabla^2\delta\varphi_{1K}\varphi_{0K}' - 6\mathcal{H}\varphi_{0K}'(\delta\varphi_{1K}'-Y_K) - \frac{8\pi G}{\mathcal{H}}\varphi_{0K}'\Big(X_M\varphi_{0K}'+a^2U_{,K}\varphi_{0M}'\Big)\delta\varphi_{1M} \\
&+2a^2U_{,M}Y_M + \frac{4\pi G}{\mathcal{H}}\Big(a^2\varphi_{0K}'^2\delta U_1 + 2a^2UY_K\varphi_{0K}' - \sum_L\varphi_{0K}'^2\varphi_{0L}'\delta\varphi_{1L}'\Big)\Bigg]\Big(\varphi_{0N}'(\delta\varphi_{1N}'-Y_N)+a^2\delta U_1\Big) \\
&+\frac{(5+3c_{\rm s}^2)}{9(\sum_P\varphi_{0P}'^2)^2}\sum_{K,L}\Big[\varphi_{0K}'\varphi_{0L}'(\delta\varphi_{1K}'-Y_K)(\delta\varphi_{1L}'-Y_L) + a^4\delta U_1^2 + 2a^2\delta U_1\varphi_{0K}'(\delta\varphi_{1K}'-Y_K)\Big] \\
&-\frac{4\pi G}{3\mathcal{H}^2\sum_P\varphi_{0P}'^2}\sum_{K,L}\Big(\varphi_{0K}'\delta\varphi_{1K,l}' - \varphi_{0K}'Y_{K,l} + a^2\delta U_{1,l}\Big)\nabla^{-2}(X_L\delta\varphi_{1L}+\varphi_{0L}'\delta\varphi_{1L}')_{,}{}^{l} \\
&-\sum_{K,L}\frac{\varphi_{0K}'\varphi_{0L}'}{18\mathcal{H}^2\sum_P\varphi_{0P}'^2}\nabla^{-2}\Bigg\{\nabla^2\left(\delta\varphi_{1K}'-Y_K+\frac{a^2\delta U}{\varphi_{0K}'}\right)\nabla^2\left(\delta\varphi_{1L}'-Y_L+\frac{a^2\delta U}{\varphi_{0L}'}\right) \\
&-\left(\delta\varphi_{1K}'-Y_K+\frac{a^2\delta U}{\varphi_{0K}'}\right)_{,ik}\left(\delta\varphi_{1L}'-Y_L+\frac{a^2\delta U}{\varphi_{0L}'}\right)_{,}{}^{ik} \\
&-24\pi G a\left[\left(\delta\varphi_{1K}'-Y_K+\frac{a^2\delta U}{\varphi_{0K}'}\right)_{,}{}^{(i}\nabla^{-2}\left(\frac{X_L\delta\varphi_{1L}}{\varphi_{0L}'}+\delta\varphi_{1L}'\right)_{,}{}^{j)}\right]_{,ij}\Bigg\}
\end{aligned}
\tag{3.15}
$$

where we have introduced, for ease of presentation, the quantity

$$
Y_K = \varphi_{0K}'\frac{4\pi G}{\mathcal{H}}\sum_L\varphi_{0L}'\delta\varphi_{1L}\,, \tag{3.16}
$$

and the adiabatic sound speed is given by (see, for example, Refs. [28, 29])

$$
c_{\rm s}^2 = 1 + \frac{2}{3\mathcal{H}}\frac{\sum_K U_{,\varphi_K}\varphi_{0K}'}{\sum_L\varphi_{0L}'^2}\,. \tag{3.17}
$$

Thus, we have achieved our goal of obtaining the expression for ζ_2 solely in terms of scalar field variables. While this may appear cumbersome, we should stress that this is valid for multiple scalar fields and, for the ease of numerical computation, we have reduced the expression to include only scalar field fluctuations and their first time derivative. Although in this paper we have presented our results in real space, this can be readily translated into Fourier space using, for example, the methods described in Ref. [24].

4. Single field, slow roll approximation

In this section, we will assume a single field. The slow roll approximation then allows us to relate the scalar field's potential and its derivative to the Hubble parameter and background field derivative through

$$2\mathcal{H}\varphi_0' \simeq -a^2 U_{,\varphi}\,, \tag{4.1}$$

and

$$\mathcal{H}^2 \simeq \frac{8\pi G}{3} a^2 U\,, \tag{4.2}$$

while $c_{\rm s}^2 = -1$ and $X = \mathcal{H}\varphi_0'$. Using these relationships, we can simplify Eq. (3.15) to give

$$\begin{aligned}
\zeta_2 = &\frac{1}{3\varphi_0'}\left[\delta\varphi_2{}' - 2\mathcal{H}\delta\varphi_2 - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_2\right] + \frac{1}{3\varphi_0'^2}\left(\delta\varphi_1{}'^2 + a^2 U_{,\varphi\varphi}\delta\varphi_1{}^2 + \delta\varphi_{1,}{}^{l}\delta\varphi_1{}'_{,l}\right)\\
&+\frac{4}{9\varphi_0'^2}\left\{\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1\right)^2 + 9\mathcal{H}^2\delta\varphi_1{}^2 - 6\mathcal{H}\delta\varphi_1\left(\delta\varphi_1 - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1\right)\right\}\\
&+\frac{4\pi G}{3\mathcal{H}}\left[\frac{8\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1{}^2 - 4\delta\varphi_1\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}^2}\varphi_0'\nabla^{-2}\left(\delta\varphi_{1,kl}\nabla^{-2}\delta\varphi_{1,}{}'^{kl} - \nabla^2\delta\varphi_1\delta\varphi_1{}'\right)\right.\\
&\left.-2\nabla^{-2}\left(\delta\varphi_1{}'\nabla^2\delta\varphi_1 + \delta\varphi_{1,l}\delta\varphi_{1,}{}^{l}\right) - \frac{1}{\mathcal{H}\varphi_0'}\nabla^{-2}\delta\varphi_1{}'_{,l}\left(\delta\varphi_1{}'_{,l} - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_{1,l} - 3\mathcal{H}\delta\varphi_{1,l}\right)\right]\\
&+\frac{2}{9\mathcal{H}\varphi_0'^2}\left[\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right]\left[\nabla^2\delta\varphi_1 - 6\mathcal{H}\delta\varphi_1{}' + 24\pi G\varphi_0'^2\delta\varphi_1 - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1{}'\right]\\
&-\frac{2}{3\varphi_0'}\delta\varphi_{1,l}\left[\frac{\delta\varphi_{1,l}}{\varphi_0'} - \frac{4\pi G}{\mathcal{H}}\varphi_0'\nabla^{-2}\delta\varphi_{1,}{}^{l\prime}\right] - \frac{1}{18\mathcal{H}^2}\nabla^{-2}\left\{\left[\nabla^2\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right)\right]^2\right.\\
&-\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right)_{,ij}\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right)_{,}{}^{ij}\\
&\left.-24\pi G\varphi_0'{}^2\left[\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right)_{,}{}^{(i}\nabla^{-2}\delta\varphi_{1,}{}'^{j)}\right]_{,ij}\right\}.
\end{aligned} \tag{4.3}$$

Taking the large scale limit of this expression, where $\nabla^2 \to 0$, gives

$$\begin{aligned}
\zeta_2 = &\frac{1}{3\varphi_0'}\left[\delta\varphi_2{}' - 2\mathcal{H}\delta\varphi_2 - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_2\right] + \frac{1}{3\varphi_0'^2}\left(\delta\varphi_1{}'^2 + a^2 U_{,\varphi\varphi}\delta\varphi_1{}^2\right) + \frac{8\pi G}{3\mathcal{H}}\nabla^{-2}\delta\varphi_{1,}{}^{l\prime}\delta\varphi_{1,l}\\
&+\frac{4}{9\varphi_0'^2}\left\{\left(\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1\right)^2 + 9\mathcal{H}^2\delta\varphi_1{}^2 - 6\mathcal{H}\delta\varphi_1\left(\delta\varphi_1 - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1\right)\right\}\\
&+\frac{4\pi G}{3\mathcal{H}}\left[\frac{8\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1{}^2 - 4\delta\varphi_1\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}^2}\varphi_0'\nabla^{-2}\left(\delta\varphi_{1,kl}\nabla^{-2}\delta\varphi_{1,}{}'^{kl} - \nabla^2\delta\varphi_1\delta\varphi_1{}'\right)\right.\\
&\left.-2\nabla^{-2}\left(\delta\varphi_1{}'\nabla^2\delta\varphi_1 + \delta\varphi_{1,l}\delta\varphi_{1,}{}^{l}\right) - \frac{1}{\mathcal{H}\varphi_0'}\nabla^{-2}\delta\varphi_1{}'_{,l}\left(\delta\varphi_1{}'_{,l} - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_{1,l} - 3\mathcal{H}\delta\varphi_{1,l}\right)\right]\\
&+\frac{2}{9\mathcal{H}\varphi_0'^2}\left[\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1 - 3\mathcal{H}\delta\varphi_1\right]\left[24\pi G\varphi_0'^2\delta\varphi_1 - 6\mathcal{H}\delta\varphi_1{}' - \frac{4\pi G}{\mathcal{H}}\varphi_0'^2\delta\varphi_1{}'\right].
\end{aligned} \tag{4.4}$$

5. Discussion

In this article, we have focused on the curvature perturbation in the uniform density gauge, ζ, at second order in perturbation theory. We have presented, for the first time, the expression for ζ_2 in terms of scalar fields, computed using full cosmological perturbation theory. Using the relevant Einstein field equations, we have replaced all metric perturbations, and so the resultant expression contains only the scalar field, its fluctuations and single time derivatives.

This expression will likely be useful for future numerical computations of inflationary observables, such as the bispectrum; we have presented the final expression in such a way to make the numerical implementation as simple as possible. Additionally, we have presented the case for a single, slowly-rolling scalar field, and have then taken the large scale limit of this expression. We can see the benefit of using full perturbation theory from this expression, Eq. (4.4), where we obtain terms containing inverse Laplacians, which would not be present in the corresponding expression from the δN approach. These terms can be removed, on assuming that the metric perturbation B is decaying at both first and second order, allowing one to use the momentum equation to replace the inverse Laplacian terms. However, this is only the case on scales much larger than the horizon. We present the more general equation, valid on all scales, in this work.

While this project was in its final stages we became aware of a study obtaining similar results, where comparison is possible [30].

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Appendix A. Second order Klein-Gordon Equation

In this section, for completeness, we present the Klein-Gordon equation for the second order scalar field, in a close form, from Ref. [24].

$$
\delta\varphi_{2I}'' + 2\mathcal{H}\delta\varphi_{2I}' - \nabla^2\delta\varphi_{2I} + a^2\sum_K\left[U_{,\varphi_K\varphi_I} + \frac{8\pi G}{\mathcal{H}}\left(\varphi_{0I}'U_{,\varphi_K} + \varphi_{0K}'U_{,\varphi_I} + \varphi_{0K}'\varphi_{0I}'\frac{8\pi G}{\mathcal{H}}U_0\right)\right]\delta\varphi_{2K}
$$
$$
+\frac{16\pi G}{\mathcal{H}}\left[\;\delta\varphi_{1I}'\sum_K X_K\delta\varphi_{1K} + \sum_K\varphi_{0K}'\delta\varphi_{1K}\sum_K a^2U_{,\varphi_I\varphi_K}\delta\varphi_{1K}\right]
$$

$$
\begin{aligned}
&+\left(\frac{8\pi G}{\mathcal{H}}\right)^2\sum_K\varphi_{0K}'\delta\varphi_{1K}\left[a^2U_{,\varphi_I}\sum_K\varphi_{0K}'\delta\varphi_{1K}+\varphi_{0I}'\sum_K\left(a^2U_{,\varphi_K}+X_K\right)\delta\varphi_{1K}\right]\\
&-2\left(\frac{4\pi G}{\mathcal{H}}\right)^2\frac{\varphi_{0I}'}{\mathcal{H}}\sum_K X_K\delta\varphi_{1K}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)+\frac{4\pi G}{\mathcal{H}}\varphi_{0I}'\sum_K\delta\varphi_{1K}'^2\\
&+a^2\sum_{K,L}\left[U_{,\varphi_I\varphi_K\varphi_L}+\frac{8\pi G}{\mathcal{H}}\varphi_{0I}'U_{,\varphi_K\varphi_L}\right]\delta\varphi_{1K}\delta\varphi_{1L}+F\left(\delta\varphi_{1K}',\delta\varphi_{1K}\right)=0\,,
\end{aligned}
\tag{A.1}
$$

where $F\left(\delta\varphi_{1K}',\delta\varphi_{1K}\right)$ contains gradients and inverse gradients quadratic in the field fluctuations and is defined as

$$
\begin{aligned}
F\left(\delta\varphi_{1K}',\delta\varphi_{1K}\right)=&\left(\frac{8\pi G}{\mathcal{H}}\right)^2\delta\varphi_{1I,l}'\nabla^{-2}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)_{,}{}^{l}-\frac{16\pi G}{\mathcal{H}}\nabla^2\delta\varphi_{1I}\sum_K\varphi_{0K}'\delta\varphi_{1K}\\
&+2\frac{X_I}{\mathcal{H}}\left(\frac{4\pi G}{\mathcal{H}}\right)^2\nabla^{-2}\left[\sum_K\varphi_{0K}'\delta\varphi_{1K,lm}\nabla^{-2}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)_{,}{}^{lm}\right.\\
&\qquad\left.-\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)\nabla^2\sum_K\varphi_{0K}'\delta\varphi_{1K}\right]\\
&+\frac{4\pi G}{\mathcal{H}}\left[\varphi_{0I}'\sum_K\delta\varphi_{1K,l}\delta\varphi_{1K,}{}^{l}+4X_I\nabla^{-2}\sum_K\left(\delta\varphi_{1K}'\nabla^2\delta\varphi_{1K}+\delta\varphi_{1K}'{}_{,l}\delta\varphi_{1K,}{}^{l}\right)\right]\\
&+\left(\frac{4\pi G}{\mathcal{H}}\right)^2\frac{\varphi_{0I}'}{\mathcal{H}}\left[\nabla^{-2}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)_{,lm}\nabla^{-2}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)_{,}{}^{lm}\right.\\
&\qquad\left.-\sum_K\varphi_{0K}'\delta\varphi_{1K,l}\sum_K\varphi_{0K}'\delta\varphi_{1K,}{}^{l}\right]\\
-\frac{\varphi_{0I}'}{\mathcal{H}}\nabla^{-2}&\left\{8\pi G\sum_K\left(\delta\varphi_{1K,l}\nabla^2\delta\varphi_{1K,}{}^{l}+\nabla^2\delta\varphi_{1K}\nabla^2\delta\varphi_{1K}+\delta\varphi_{1K}'\nabla^2\delta\varphi_{1K}'+\delta\varphi_{1K}'{}_{,l}\delta\varphi_{1K}'{}_{,}{}^{l}\right)\right.\\
&\left.-\left(\frac{4\pi G}{\mathcal{H}}\right)^2\left[2\nabla^{-2}\sum_K\left(X_K\delta\varphi_{1K}+\varphi_{0K}'\delta\varphi_{1K}'\right)_{,}{}^{i}{}_{j}\sum_K X_K\delta\varphi_{1K}+\sum_K\varphi_{0K}'\delta\varphi_{1K,}{}^{i}\sum_K\varphi_{0K}'\delta\varphi_{1K,j}\right]_{,i}^{\;j}\right\}
\end{aligned}
\tag{A.2}
$$

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